Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 8

Exercise 1 *(4 Points)*
Let $\mu_\lambda(dx)$ be the Poisson distribution on $\mathbb{R}$ with parameter $\lambda > 0$, i.e.

$$\mu_\lambda(dx) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_k(dx)$$

(a) Show that $\Gamma = \{\mu_\lambda : \lambda \in K\}$ is relatively compact, where $K \subset (0, \infty)$ is bounded and closed. Are all limit points of Γ already contained in Γ?

(b) Does there exists $\mu_0, \mu_\infty \in \mathcal{P}(\mathbb{R})$ such that $\mu_\lambda \to \mu_0$ as $\lambda \to 0$ or $\mu_\lambda \to \mu_\infty$ as $\lambda \to \infty$ weakly?

Exercise 2 *(4 Points)*
Let μ_λ be the exponential distribution with parameter $\lambda > 0$, i.e.

$$\mu_\lambda(dx) = f_\lambda(x)dx,$$

where $f_\lambda(x) = \mathbb{1}_{[0,\infty)}(x)\lambda e^{-\lambda x}$, $x \in \mathbb{R}$.

(a) Show that $\Gamma = \{\mu_\lambda : \lambda \in K\}$ is relatively compact, where $K \subset (0, \infty)$ is compact. Are all limit points of Γ already contained in Γ?

(b) Does there exists $\mu_0, \mu_\infty \in \mathcal{P}(\mathbb{R})$ such that $\mu_\lambda \to \mu_0$ as $\lambda \to 0$ or $\mu_\lambda \to \mu_\infty$ as $\lambda \to \infty$ weakly?

Exercise 3 *(4 Points)*
Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$. Prove that μ_n is tight if and only if the respective sequence of distribution functions $F_n(t) := \mu_n((-\infty, t])$ satisfies

$$\lim_{t\to\infty} \sup_{n\geq 1} |F_n(t) - 1| = 0, \quad \text{and} \quad \lim_{t\to-\infty} \sup_{n\geq 1} |F_n(t)| = 0.$$

Exercise 4 *(4 Points, talk)*
Explain the main steps in the proof of Prokhorov's theorem.